Resonant effect of plasma waves on charged particles in a magnetic field

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Resonances are discussed between electrons and protons and waves of whistler and hydromagnetic types. The conditions of resonance are investigated. Approximations for the effects on the particles are obtained. The occurrence of a 'trapping' phenomenon is demonstrated and the effect on velocity distributions is discussed.

1. Introduction

An important simplification in the very complicated subject of collision-free plasma results from the fact that, when the varying part of the field is represented by waves, a charged particle with a particular velocity interacts much more strongly with waves of certain frequencies than with others. A number of workers (for a review see Vedenov, Velikhov & Sagdeev 1961) have studied the dispersion equation for a uniform plasma in a uniform magnetic field. Then the perturbation in the velocity distribution is found to have singularities at the velocities at which the frequency seen by the particles is an integral multiple of their gyro-frequency, and this is the resonance condition. Here the effect of a wave on an individual particle is studied, the motion of the particle through the wave field being included in the calculations.

One restriction is imposed on the type of wave considered, which is related to the possibility of working in a frame of reference in which the wave is static. Changes in the frame of reference are restricted to motion parallel to the unperturbed magnetic field **B**, because motion across this field would introduce a uniform electric field. If the wave normal makes an angle θ with **B** and the phase velocity is w, the existence of such a frame in which the wave is static requires $w < c\cos\theta$. This is assumed, and is not a very severe restriction, since the waves studied later in the paper are slow. If $w > c\cos\theta$, a relativistic transformation is possible to a frame in which the wave normal is perpendicular to **B**, and the effect of such waves on particles is of a different kind.

The analysis presented here depends on the unperturbed state being uniform and is restricted to the effect of a single sinusoidal wave. The limitations of this formulation are discussed towards the end. Even with these limitations it is a sizable task to cover the whole spectrum of waves, and the intention here is mainly to sort out the possibilities, leaving a number of them for later exploration.

2. First-order effect on a particle

The first-order effect on a particle is obtained simply by integrating along the unperturbed trajectory which is a helix

$$x = a\cos\Omega t, \quad y = a\sin\Omega t, \quad z = v_{\parallel} t, \tag{1}$$

where the z-axis is parallel to $\mathbf{B}$, Ω is the angular gyrofrequency and $\Omega a = v_{\perp}$.

To calculate only the change in $v_{\parallel}$ we work in the frame in which the field is static and use the wave-field components

$$E_z = E\cos(\alpha+\varphi), \quad B_x = b_x\cos(\beta_x+\varphi), \quad B_y = b_y\cos(\beta_y+\varphi), \tag{2}$$

where $\varphi = ky\sin\theta + kz\cos\theta$, the wave number $\mathbf{k}$ is perpendicular to the x-axis and α, β_x and β_y are constant phase angles.

The equation of motion is, using (1),

$$(m/e)\,dv_{\parallel}/dt = E_z - (v_{\perp}/c)\,(B_y\sin\Omega t + B_x\cos\Omega t). \tag{3}$$

It is convenient to introduce $R = -ka\sin\theta$ and $N = kv_{\parallel}\cos\theta/\Omega$, N being an integer when resonance occurs. Since a is the Larmor radius and $2\pi v_{\parallel}/\Omega$ is the pitch of the helix, R and N depend on the ratio of these to the wavelength. Then using (1), $\phi = -R\sin\Omega t + N\Omega t$.

One turn of the helix corresponds to a time $2\pi/\Omega$ and to a change in phase of the wave of $2\pi N$. The changes $\delta v_{\parallel}$ in particle velocity parallel to $\mathbf{B}$ in successive turns of the helix will then be proportional to $\cos(\gamma + 2\pi Nn)$ where γ is some constant and n refers to the nth turn. Then the total change after a large number of turns is a Fourier sum and has resonances when N is an integer. The calculation of δv will now be performed for integral values of N.

We use

$$\int_0^{2\pi/\Omega} \cos(N\Omega t - R\sin\Omega t)\,dt = 2\pi\Omega^{-1}J_N(R) \tag{4}$$

and

$$\int_0^{2\pi/\Omega} \sin(N\Omega t - R\sin\Omega t)\,dt = 0.$$

Using (2)

$$B_y\sin\Omega t + B_x\cos\Omega t = b_1\cos(\phi-\Omega t) + b_{-1}\cos(\phi+\Omega t) + b_{-1}\sin(\phi-\Omega t) + b_{-1}\sin(\phi+\Omega t),$$

where $\quad b_1 = \tfrac{1}{2}(b_x\cos\beta_x + b_y\sin\beta_y), \quad b_{-1} = \tfrac{1}{2}(b_x\cos\beta_x - b_y\sin\beta_y),$

and b_1' and b_{-1}' are similar but irrelevant, and they all represent circularly polarized components. Then integrating (3) from 0 to $2\pi/\Omega$ and multiplying by $c\Omega/2\pi$ gives

$$B\delta v_{\parallel}/2\pi = cE\cos\alpha J_N(R) - v_{\perp}[b_1 J_{N-1}(R) + b_{-1}J_{N+1}(R)]. \tag{5}$$

3. The resonant velocities

The resonant values of $v_{\parallel}$ are just integral multiples of $\Omega\sec\theta/k$, but it is necessary to consider the dependence of the Bessel functions in (5) on R. The oscillatory behaviour of the Bessel functions is not so important here as the fact that Bessel functions of high order are very small when the argument is small.

When $R \ll N^{\frac{1}{2}}$, $J_N(R) \approx (R/2)^N/N!$ and, a stronger property, if $N \gg 1$, $J_N(R) \ll 1$ for any value of R appreciably less than N. For large values of N, therefore, resonance is ineffective, if R is appreciably less than N, which reduces to $v_\perp$ being appreciably less than $v_\parallel \cot\theta$ in the frame of the wave. Assuming that any of the terms in (5) can be important the pattern of velocities at which resonance is effective appears as in figure 1. The spacing between the lines is $\Omega \sec\theta/k$ and the bottoms of the lines lie approximately on a line of slope $\cot\theta$. Suppose now that

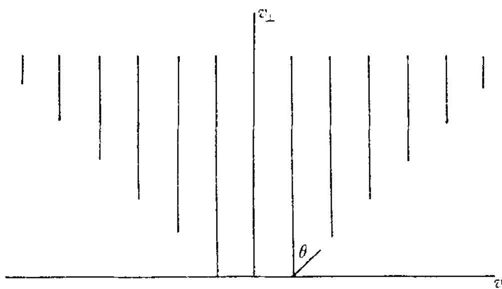

FIGURE 1. Resonant velocities in the frame of the wave.

the plasma defines a natural frame of reference in which there is no steady electric field, so that this frame is moving relative to the wave in the direction of **B**. Let the wave's angular frequency be ω and phase velocity $w = \omega/k$. In this frame the pattern of resonant velocities is shifted parallel to the $v_\parallel$ axis by $w \sec\theta$ and

$$v_\parallel = w \sec\theta(1 \pm N\Omega/\omega), \tag{6}$$

where the $\pm$ depends on the sign of the charge on the particle. The sign can most easily be checked by putting $\omega = \Omega$, $v_\parallel = 0$, $N = \pm 1$ and looking at (5); the term with J_0 will be b_1 or b_{-1} and this must rotate in the same sense as the particle concerned.

It is now necessary to consider the wave speed, which should be calculated from the dispersion equation using the velocity distribution of the plasma. The theory of the dispersion equation is available only for restricted cases, however, and tends to be untrustworthy when there is a lot of overlap between the regions of velocity space occupied by the main body of the plasma particles and the resonant velocities. Clearly this depends strongly on the value of w compared to the velocity spread of the main body of particles. If w is small the scale of figure 1 is small and many resonances are important. With regard to very slow waves then, little more can be said without further development of dispersion equations. The converse is useful, however, that, for waves which are not too slow according to known dispersion equations, the resonances should not have a very strong effect on the dispersion, because few particles are involved.

The dependence of the resonant velocities on frequency can be seen from (6). For very low frequencies, $\omega \ll \Omega$, the resonant velocities are widely spaced compared to w and it is likely that the resonance $N = 0$, corresponding to particles moving with the wave, is most important. For very high frequencies, $\omega \gg \Omega$, the resonances are closely spaced, and then the limitation that R should not be appreciably less than N is important and figure 1 shows that there is a minimum speed required for resonance of approximately w. In order to get resonance for speeds much less than w it is necessary to have $\omega \sim \Omega$, for zero speed the requirement being $\omega = \Omega$ with $N = \pm 1$. It is possible to be more specific by using an approximate dispersion equation. Dispersion equations are known to first order in the plasma temperature, but here for simplicity we use the zero temperature approximation, which is summarized in the Appendix. This treatment is well justified when there is a cool plasma and the interest is in interactions between waves and suprathermal particles, which are not sufficiently numerous to have much effect on the wave. An example of this is the interaction of waves in the exosphere with energetic particles of the Van Allen belt, where the most numerous particles belong to a hydrogen plasma at about 1000° K. It is further assumed that $B^2 \ll 4\pi n m_e c^2$, implying that Ω_e is much less than the plasma frequency. Then no modes propagate at frequencies between $\Omega_e \cos\theta$ and the plasma frequency, and attention will here be confined to lower frequencies, though there are slow waves of plasma-oscillation type near the plasma frequency. In the range $\Omega_p \ll \omega \ll \Omega_e \cos\theta$, only the whistler mode propagates and a convenient approximation is (see Appendix)

$$(w/v_A)^2 = (\omega/\Omega_p)(\cos\theta - \omega/\Omega_e), \tag{7}$$

where v_A is the Alfvén velocity. At lower frequencies $w \sim v_A$, except that when the second mode appears at $\omega \approx \Omega_p$, its velocity is very small over a narrow band of frequencies, but the very slow waves are not to be discussed. The waves with $w \sim v_A$ may be regarded as hydromagnetic. The effects on electrons and protons need separate discussion. It is convenient to express particle energies as multiples of $W = B^2/8\pi n$, the magnetic energy per electron or proton, which is also $\frac{1}{2}m_p v_A^2$.

Electrons. Because $\Omega_e > \omega$, the resonances are widely spaced. The energy required for resonance with $N = 0$ is $\simeq (\omega/\Omega_e) W \sec\theta$ for whistlers and $\simeq (m_e/m_p) W \sec^2\theta$ for hydromagnetic waves. The energy for $N = 1$ is $\simeq (\Omega_e/\omega) W \sec\theta$ for whistlers and $\simeq (\Omega_e \Omega_p/\omega^2) W \sec^2\theta$ for hydromagnetic waves, and typically the value $(m_p/m_e) W$ (for $\omega = \Omega_p \sec\theta$) is rather a large energy.

Protons. For $\omega < \Omega_p$, the story is similar to that for electrons. $N = 0$ requires an energy $\simeq W \sec^2\theta$ and $N = 1$ requires one $\simeq (\Omega_p/\omega)^2 W \sec^2\theta$, these energies being respectively larger and smaller than the corresponding values for electrons.

For $\omega \sim \Omega_p$, very slow protons can resonate with $N = 1$, as already mentioned, but this will not be further pursued.

For $\omega \gg \Omega_p$, the resonances are narrowly spaced. The minimum energy for resonance is $\frac{1}{2}m_p w^2$, or, since (7) is then valid, $(\omega/\Omega_p) W \cos\theta$.

The overall conclusions are that, except near special frequencies, the important resonance for electrons is $N = 0$, while protons generally need energies rather greater than W in order to resonate.

4. The velocity change

In the frame of the wave, the wave being static, the electric field has a potential which varies sinusoidally with wave phase. Then a resonant particle, returning to the same phase after each turn of its trajectory, has its energy conserved to first order, so that $v_\perp \delta v_\perp = -v_\parallel \delta v_\parallel$. In this frame therefore the direction of the 2-vector $(v_\parallel, v_\perp)$ is changed by an angle $\delta = \delta v_\parallel / v_\perp$ determined by (5). We now have to consider the relative importance of the terms in (5), using the ratios of the wave components from the Appendix.

When $\omega \gg \Omega_p$ we have (7) and

$$wb_l/cE \approx (\Omega_e/\omega)\cos\theta\,(\cot\tfrac{1}{2}\theta)^l. \tag{8}$$

For the cases $N = 0$ and 1, R can be small and it may be supposed that $J_0(R) \approx 1$ and $J_1(R) \approx \frac{1}{2}R$. Then we get

$$N = 0\colon\ \frac{cEJ_0(R)}{v_\perp b_l J_1(R)} \sim 4\left(\frac{w}{v_\perp}\right)^2 \frac{\Omega(\cot\frac{1}{2}\theta)^l}{\Omega_e \sin 2\theta},$$

$$N = 1\colon\ \frac{cEJ_1(R)}{v_\perp b_l J_0(R)} \sim \frac{\omega^2\tan\theta}{2\Omega\Omega_e\cot\frac{1}{2}\theta}.$$

For larger values of N it may be supposed that $J_N(R)$ and $J_{N-1}(R)$ are comparable and (8) multiplied by $v_\perp/w$ then gives the relevant comparison. These results apply to whistlers and in general it is seen that the electric and magnetic terms do not usually differ by an enormous factor. For frequencies below Ω_p the expression for E/b_l is more complicated (see Appendix), but in general the magnetic terms dominate. From (5) then

$$\delta \sim 2\pi(b_l/B)\,J_{N-1}(R) \tag{9}$$

or more if the electric term dominates. This is the change for one turn of the helical trajectory and the total change is obtained by multiplying by the number of turns over which the resonance is effective.

5. Particle trapping

In practise for very weak waves the limitation to the resonance will arise from non-uniformity of the unperturbed field or from the departure of the wave from sinusoidal form. In the former case the number of turns of the helix over which resonance is effective is of order $\{2\pi v_\parallel d(\Omega^{-1})/dz\}^{-\frac{1}{2}}$, which is generally larger for electrons than for protons. In the latter case the number of wave periods over which resonance is effective is of order $\{2\pi\, d(\omega^{-1})/dt\}^{\frac{1}{2}}$, in a frame following the mean motion of the particle concerned. If the wave is not too weak, non-linear effects may limit the resonance and these are discussed in this section.

In the absence of a magnetic field resonance occurs only for particles moving with the wave and it is well known that 'particle trapping' then occurs. Taking the wave normal in the z-direction, and working in the frame of the wave $\dot{x}$, $\dot{y}$ and $\frac{1}{2}m\dot{z}^2 + e\phi\cos kz$ are constant, $\phi\cos kz$ being the potential. Then if $\dot{z}$ is small enough z oscillates in a potential trough, out of which it can never get. The largest possible change in $\dot{z}$ is then $4(e\phi/m)^{\frac{1}{2}}$. It will now be shown that a similar trapping phenomenon occurs in the presence of a magnetic field.

Suppose that in one turn of the helix a resonant particle suffers displacements from its unperturbed trajectory δz and $\delta\mathbf{r}_\perp$ and velocity changes $\delta v_\parallel$ and $\delta\mathbf{v}_\perp$, which are small. In the next turn of the helix the changes are almost the same except for z which changes by $\delta z + 2\pi\delta v_\parallel/\Omega$. There is no similar change in $\delta\mathbf{r}$

because $\mathbf{v}_\perp$ is continually rotating in the magnetic field. Assuming a small enough amplitude the changes after n turns are $n\delta z + \pi n(n-1)\,\delta v_\parallel/\Omega$, $n\delta r_\perp$, $n\delta v_\parallel$ and $n\delta v_\perp$. If this is valid for a large enough number of turns the term in n^2 must be the dominant non-linear effect. The change in z changes the phase of the particle in the wave, and this of course changes the values of all the changes per turn. It is now assumed that only this phase change is important and that the change in $\delta v_\parallel$ due to changes in $v_\perp$, R and N is negligible because they are of first order in n. This of course depends on n being large and requires that the unperturbed field be nearly uniform and the wave nearly sinusoidal. Let the resonant velocity concerned be v_r and let $(\Delta v = v_\parallel - v_r$, and $\Delta z = z - v_r t$, so that $d\Delta z/dt = \Delta v_\parallel$. Now the change of phase, which has been assumed to be the only important effect, can be expressed in the form

$$\delta v_\parallel = A\cos(\alpha + kz\cos\theta),$$

and, since $2\pi k v_r \cos\theta/\Omega$ is an integer for resonance, the values of $\delta v_\parallel$ for successive turns of the helix follow a relation of the form

$$\delta v_\parallel = A\cos(\alpha' + k\Delta z\cos\theta).$$

The average motion over many turns of the spiral may be approximated by a corresponding continuous variation

$$dv_\parallel/dt = (\Omega A/2\pi)\cos(\alpha' + k\Delta z\cos\theta),$$

and this can be integrated after multiplication by $\Delta v_\parallel$ to give

$$(\Delta v_\parallel)^2 = C + (\Omega A/\pi k\cos\theta)\sin(\alpha' + k\Delta z\cos\theta),$$

where C is a constant greater than $-\Omega A/\pi k\cos\theta$. This shows that particles near enough to resonance oscillate about the velocity of exact resonance just as in the case of trapping with no magnetic field. The condition for trapping is $C < \Omega A/\pi k\cos\theta$. The largest possible change in $v_\parallel$ is $2(2\Omega A/\pi k\cos\theta)^{\frac{1}{2}}$ and, taking the order of magnitude of A to be the same as that of $\delta v_\parallel$ given by (9), this gives $4v_\perp\{b_l J_{N-1}(R)\tan\theta/BR\}^{\frac{1}{2}}$ and the number of turns of the helix before the non-linear effect is important is

$$\{R\cot\theta(b_l/B)\,J_{N-1}(R)\}^{-\frac{1}{2}}.$$

6. Effect on the velocity distribution

If collisions are negligible, Liouville's theorem can be used to find the effect of a wave on the velocity distribution from its effect on individual particles. The velocity distribution is in principle a function of six variables in addition to the time, but fortunately these can be reduced. The unperturbed state being uniform the position of the particle enters only through the phase of the wave. The co-ordinate perpendicular to $\mathbf{B}$ and $\mathbf{k}$ is not involved at all, and, if a variable ψ is introduced to represent the phase of the wave relative to the unperturbed helical trajectory of the particle, the other co-ordinates do not appear explicitly. We revert now to the frame of the wave, because then the energy of the particle is constant to first order. The unperturbed distribution is assumed to be symmetrical about the direction of $\mathbf{B}$, so that the angle representing rotation of the velocity about $\mathbf{B}$ enters only into the phase ψ. Consequently the relevant variables reduce to the phase ψ and one other, which must involve $v_{\parallel}$ or $v_{\perp}$, and it is convenient to choose $v_{\parallel}$. It is now possible to draw diagrams. The previous section has shown that the non-linear effect enters by way of $v_{\parallel}$ and near a resonance we can put

$$d\psi/dt = k\cos\theta\,(v_{\parallel} - v_r),$$

where v_r is the velocity for exact resonance. $dv_{\parallel}/dt$ of course varies sinuosidally with ψ and is proportional to the amplitude of the wave. The problem is then the

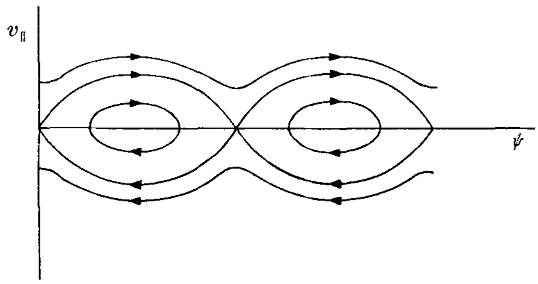

FIGURE 2. Trajectory of a particle.

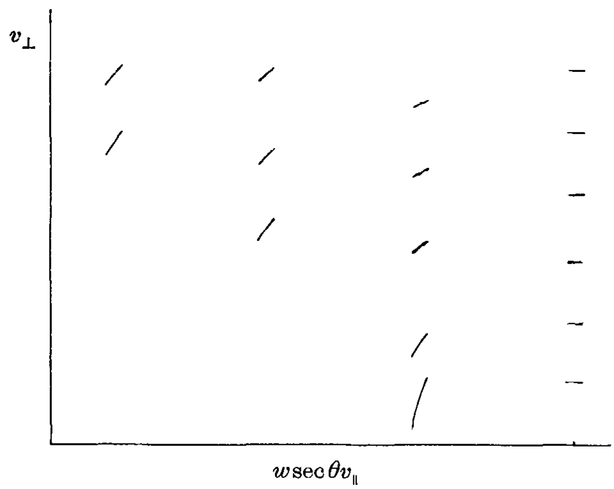

FIGURE 3. The change in the distribution function.

same as in the absence of a magnetic field. The trajectory of a particle on a plot of $v_{\parallel}$ against ψ is shown in figure 2, and is essentially the same as in Dungey (1961). The argument used there then shows that the effect of the wave is to reduce the rate of change of the distribution function with velocity to a very small value in the neighbourhood of v_r. Here it must be remembered that the energy is constant and it is the rate of change of distribution function with the direction of $\mathbf{v}$ in the frame of the wave that is reduced. This is indicated in figure 3. It may be said that after the wave has become effective, the contours of constant distribution function must follow the arcs in figure 3 in the resonant bands. Some qualitative comments may be made and it is convenient to return to the frame of the plasma.

(i) The $N = 0$ resonance does not change $v_\perp$ and, if the distribution function decreases with $v_\parallel$, this leads to an increase in the total energy of motion parallel to the field.

(ii) For particles with velocities much greater than the wave, the main effect is to smear out anisotropy of the velocity distribution.

(iii) If there is a resonance near $v_\parallel = 0$ (which from (6) is impossible for $\omega < \Omega$) there is a tendency to increase the energy of motion perpendicular to the field.

I am indebted to Dr F. D. Kahn for a very valuable discussion, in which he pointed out the error in an earlier attack.

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Appendix

Waves in a cold plasma

The theory of waves in a cold plasma, though well known, is outlined here for reference.

Working in the frame of the plasma, the velocity of the particles $\mathbf{v}$ is only that due to the wave. Taking the wave variables to behave like

$$\exp i(\omega t + ky \sin\theta + ks\cos\theta)$$

it is convenient to use the notation

$$v_1 = v_x + iv_y, \quad v_{-1} = v_x - iv_y, \quad v_0 = v_z,$$

and the suffix l for 0, ± 1. Then

$$i(\omega \pm l\Omega)\, v_l = (e/m)\, E_l,$$

where the $\pm$ refers to the sign of the particle.

The current density j_l reduces to

$$4\pi i j_l = \omega_p^2\, \omega E_l/(\omega - l\Omega_e)\,(\omega + l\Omega_p),$$

where ω_p is the plasma frequency. The result of substituting this into Maxwell's equations may be written

$$\{1 - (w/c)^2 + (w/v_A)^2\,[(l - \omega/\Omega_e)\,(l + \omega/\Omega_p)]^{-1}\}\, E_l = (\hat{\mathbf{k}}\,.\,\mathbf{E})\,\hat{\mathbf{k}}_l, \tag{A}$$

where v_A is the Alfvén velocity, $w = \omega/k$ and $\hat{\mathbf{k}}$ is the unit vector parallel to k. Putting $k_0 = \cos\theta$ and $k_{\pm 1} = \pm i \sin\theta$, the dispersion equation is obtained by eliminating $\mathbf{E}$ from (A). Apart from $w = 0$, which is a root for all values of ω, the dispersion equation is a quadratic for w^2. After much algebra it can be written

$$\xi^2 + \xi \sin^2\theta - (\omega/\Omega_p - \omega/\Omega_e)^2 \cos^2\theta = (\xi + 1)\{\xi^2 - (\omega/\Omega_p - \omega/\Omega_e)^2\}(v_A/c)^2\{1 - (w/c)^2\}, \tag{B}$$

where $\xi = (w/v_A)^2\,(1 - w^2/c^2)^{-1} - 1 + \omega^2/\Omega_e\,\Omega_p$. (B) is convenient, if the right-hand side is negligible, as is the case here. The factor $(v_A/c)^2$ is small and the right-hand side of (B) is important only if $\xi(v_A/c)^2$ is not small. Here the interest is in waves with $\omega < \Omega_e$ and w comparable to the particle speeds, so that $(w/c)^2$ is small and the factor $(1-w^2/c^2)$ in the definition of ξ can also be dropped. Usually $B^2 \ll 4\pi n m_e c^2$ and then $\Omega_e \ll \omega_p$. In this case there are no modes of propagation at frequencies between Ω_e and ω_p. Waves of frequency greater than ω_p either have phase velocities faster than light or have the nature of plasma oscillations. The latter can resonate with particles, but here only waves of frequency less than Ω_e will be discussed. Neglect of the factor $(1-w^2/c^2)$, which is equivalent to omitting the displacement current, is then justified.

When $\omega \gg \Omega_p$, the protons have little effect on the wave. The corresponding approximation for the positive root of (B) may be written

$$(w/v_A)^2 = (\omega/\Omega_p)(\cos\theta - \omega/\Omega_e). \tag{7}$$

There are two critical frequencies at which $w = 0$, which will be referred to as $\omega_1 \approx \Omega_e \cos\theta$ and $\omega_2 \approx \Omega_p$. Between ω_1 and ω_2 the mode of propagation is the whistler mode. Below ω_2 there are two modes of propagation, which are essentially hydromagnetic. When $\omega \ll \Omega_p$, the values of w are approximately v_A and $v_A \cos\theta$ corresponding to the 'fast' and 'transverse' waves.

The polarisation of the wave may be obtained from (A). For our application the magnetic components b_1 and b_{-1} are required. From $i\omega\mathbf{b} = ic\mathbf{k} \wedge \mathbf{E}$,

$$\frac{wb_l}{cE_0} = -\frac{\sin\theta\{\Omega_e\Omega_p/\omega^2 + l(\Omega_e - \Omega_p)/\omega\}}{1-(v_A/w)^2(l-\omega/\Omega_e)(l+\omega/\Omega_p)}. \tag{C}$$

In the middle of the whistler range, using (7), (C) may be approximated by

$$wb_l/cE_0 = (\Omega_e/\omega)\cos\theta(\cot\tfrac{1}{2}\theta)^l. \tag{8}$$

For application the field components are needed in the frame moving with the wave. However, E_0 is changed only by a relativistic factor and

$$\omega b_l' = ilc\{k_l E_0 - (1-w^2/c^2)k_0 E_l\}.$$

The difference between b_l' and b_l is significant only when b_l is small, which rather curiously occurs when w is small.